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$$\text{Volume } C\text{-}APBQ = V = \int_r^R S\,dy = \frac{h}{R-r}\int_r^R \left(x^2 \operatorname{arc\,cos} \frac{2Rr - (R+r)x}{(R-r)x} \right.$$

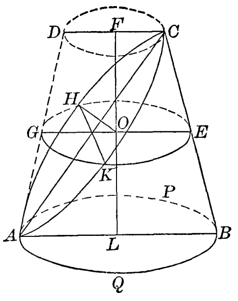
$$\left. - \frac{1}{(R-r)^2}[2Rr - (R+r)x]\sqrt{4Rr(R+r)x - 4Rrx^2 - 4R^2r^2} \right) dx,$$

in which, area $HKE = S$, and

$$dy = \left(\frac{h}{R-r}\right) dx;$$

since

$$y = \frac{h(x-r)}{R-r}.$$



Integrating, we obtain

$$V = \frac{\pi h R^{\frac{3}{2}}}{3(R-r)}\left(R^{\frac{3}{2}} - r^{\frac{3}{2}}\right).\text{[1]}$$

Putting $h = 10$, $R = 4$, and $r = 3$, we get

$$V = \frac{80}{3}(8 - 3\sqrt{3})\pi = 234.895 \text{ cu. in.}$$

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the equation

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

SOLUTION BY O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Let

$$z = \frac{u}{(x+y)^2}. \tag{1}$$

Then, computing $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial x \partial y$, and substituting in the given equation, we may write the result as follows,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{2u}{(x+y)^2} = 0.$$

Now let

$$\frac{\partial u}{\partial x} - \frac{u}{x+y} = v. \tag{2}$$

Then

$$u = \frac{(x+y)^2}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2} v. \tag{3}$$

Computing $\partial u/\partial x$ and $u/(x+y)$, and substituting in (2), we have

$$\frac{\partial u}{\partial x} - \frac{u}{x+y} = \frac{(x+y)^2}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{x+y}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2}\frac{\partial v}{\partial x} = v;$$

or

$$\frac{\partial^2 v}{\partial x \partial y} + \frac{1}{x+y}\frac{\partial v}{\partial y} - \frac{1}{x+y}\frac{\partial v}{\partial x} - \frac{2v}{(x+y)^2} = 0.$$

This equation may be written in the form,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) = 0.$$

Finally, let

$$\frac{\partial v}{\partial x} + \frac{v}{x+y} = w. \tag{4}$$

Then

$$\frac{\partial w}{\partial y} - \frac{w}{x+y} = 0; \qquad \text{or} \qquad \frac{1}{x+y}\frac{\partial w}{\partial y} - \frac{w}{(x+y)^2} = 0,$$

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[1] See Finkel's *Solution Book*, p. 319.

which may be written

$$\frac{\partial}{\partial y}\left(\frac{w}{x+y}\right) = 0, \text{ or on integrating, } \frac{w}{x+y} = \phi'''(x),$$

$\phi'''(x)$ being an arbitrary function of x.

Substituting the value of w thus given in (4), we find

$$(x+y)\frac{\partial v}{\partial x} + v = (x+y)^2\phi'''(x), \qquad \text{or} \qquad \frac{\partial}{\partial x}[(x+y)v] = (x+y)^2\phi'''(x).$$

Integrating with respect to x, the right-hand member being integrated twice by parts, we find

$$(5) \qquad (x+y)v = (x+y)^2\phi''(x) - 2(x+y)\phi'(x) + 2\phi(x) + 2\psi(y),$$

$\psi(y)$ being an arbitrary function of y with the factor 2 included for symmetry.

From (3), we have

$$(x+y)\frac{\partial v}{\partial y} - v = \frac{2u}{x+y}.$$

If we differentiate (5) with respect to y, we obtain

$$(x+y)\frac{\partial v}{\partial y} + v = 2(x+y)\phi''(x) - 2\phi'(x) + 2\psi'(y).$$

By substituting

$$(x+y)\frac{\partial v}{\partial y} = v + \frac{2u}{x+y},$$

we find

$$2v + \frac{2u}{x+y} = 2(x+y)\phi''(x) - 2\phi'(x) + 2\psi'(y);$$

or

$$(x+y)v + u = (x+y)^2\phi''(x) - (x+y)\phi'(x) + (x+y)\psi'(y).$$

Substituting the value of $(x+y)v$ from (5), we obtain

$$u = (x+y)[\phi'(x) + \psi'(y)] - 2\phi(x) - 2\psi(y).$$

Hence, from (1),

$$z = \frac{\phi'(x) + \psi'(y)}{x+y} - \frac{2\phi(x) + 2\psi(y)}{(x+y)^2};$$

or

$$z = (x+y)\left[\frac{\partial}{\partial x}\frac{\phi(x)}{(x+y)^2} + \frac{\partial}{\partial y}\frac{\psi(y)}{(x+y)^2}\right].$$

412. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Given a triangular field of sides a, b, and c. Show how to divide the field into two equal parts by a straight fence so that the cost of the fence is the least.

Solution by W. R. Ransom, Tufts College.

Suppose $a > b > c$. Connect points on the sides a and b so as to form a new triangle with the same included angle and sides of length xa and yb. If this triangle has half the area of the given triangle, $y = 1/2x$. The square of length of the fence is then

$$\varphi(x) = a^2x^2 + \frac{b^2}{4x^2} - (a^2 + b^2 - c^2).$$

The minimum value of $\phi(x)$ is found to be $\frac{1}{2}[c^2 - (a-b)^2]$, where $x = \sqrt{b/2a}$ and $y = \sqrt{a/2b}$. From $a < b + c$ and $c < b$ we get $a < 2b$; hence $y < 1$. Similarly, $x < 1$ so that this fence lies wholly within the given field. Moreover, this fence is less than the minimum fence built across either of the other corners; for, multiply $a < b + c$ by $2(b - c)$ which is positive, and add $c^2 - b^2$, then, $c^2 - b^2 + 2ab - 2ac < b^2 - c^2$, or $c^2 - (b-a)^2 < b^2 - (c-a)^2$. Similarly, interchanging a and b, $c^2 - (b-a)^2 < a^2 - (c-b)^2$. This minimum triangle is isosceles, the sides along a and b being each $= \sqrt{ab/2}$.